

# ON BANACHIC KERNELS AND APPROXIMATION THEORY

MARC ATTEIA

ABSTRACT. In this paper, I generalize a previous one about hilbertian kernels and approximation theory

## 1. DEFINITION OF A BANACHIC KERNEL.

Let  $\mathcal{E}$  be a locally convex vectorial space (lcs),  $\mathcal{E}'$  its topological dual and  $\langle \cdot, \cdot \rangle$  their duality bracket.

**1.1. Hilbertian kernel.** Let us suppose that  $\mathcal{H}$  is a subspace of  $\mathcal{E}$  and  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  a hilbertian subspace of  $\mathcal{E}$ .

Then :  $\mathcal{H} \subset \mathcal{E}$  and the injection  $j$  from  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  into  $\mathcal{E}$  is continuous.

Let  $\Lambda$  the duality mapping from  $\mathcal{H}$  into  $\mathcal{H}'$ .

The mapping  $H = j\Lambda^t j$  is called the hilbertian kernel of  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  relatively to  $\mathcal{E}$ .

So, we set :  $\tilde{H} = \Lambda^t j$ .

One can prove that :

(\*)  $H$  is linear.

(\*\*)  $\tilde{H}\mathcal{E}'$  is dense in  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ .

**1.2. (Banachic) kernel of a reflexive and strictly convex Banach space.** Let  $(\mathcal{B}, \|\cdot\|)$  a reflexive and strictly convex Banach space,  $(\mathcal{B}', \|\cdot\|_*)$  and  $(\mathcal{B}'', \|\cdot\|_{**})$  respectively, its topological dual and its topological bidual.

We suppose that  $\mathcal{B}$  is a vectorial subspace of  $\mathcal{E}$  and that the injection  $j$  from  $(\mathcal{B}, \|\cdot\|)$  into  $\mathcal{E}$  is continuous.

So, we denote by :

(\*)  $\chi$  the canonical injection from  $(\mathcal{B}, \|\cdot\|)$  into  $(\mathcal{B}'', \|\cdot\|_{**})$  ;

(\*\*)  $\psi$  an increasing function from  $\mathbb{R}_+$  into  $\mathbb{R}_+$  such that :

$$\psi(0) = 0 \quad \text{and} \quad \lim_{\rho \rightarrow +\infty} \psi(\rho) = +\infty;$$

(\*\*\*)  $J$  (resp.  $J_{\psi^{-1}}^*$ ) the duality mapping from  $(\mathcal{B}, \|\cdot\|)$  into  $2^{\mathcal{B}'}$  (resp. from  $\mathcal{B}'$  into  $2^{\mathcal{B}''}$ ).

Thus :

$$(\forall x \in \mathcal{B}, J_{\psi}(x) = \{ x^* \in \mathcal{B}' ; \langle x^*, x \rangle = \|x^*\|_* \|x\| \text{ and } \|x^*\|_* = \psi(\|x\|) \})$$

Moreover we denote by  $Ban(\mathcal{E})$ , the set of all Banach spaces which are vectorial subspaces of  $\mathcal{E}$  with a continuous injection into  $\mathcal{E}$ . The kernel of  $(\mathcal{B}, \|\cdot\|)$  is the following mapping :

$$B = j \circ \chi^{-1} \circ J_{\psi^{-1}}^* \circ {}^t j$$

We set :

$$\tilde{B} = \chi^{-1} \circ J_{\psi^{-1}}^* \circ {}^t j$$

Then :

- (\*)  $B$  is a (multi-)mapping from  $\mathcal{E}'$  in  $\mathcal{E}$  , non-linear generally.
- (\*\*)  $\tilde{B}\mathcal{E}'$  is dense in  $(\mathcal{B}, \|\cdot\|)$  .
- (\*\*\*)

$$\forall x \in \mathcal{B} , \forall e' \in \mathcal{E}' , \langle jx, e' \rangle = \psi \left( \|\tilde{B}e'\| \right) \cdot \left( \frac{d}{d\lambda} \left( \|\tilde{B}e' + \lambda x\| \right) \right)_{\lambda=0}$$

### 1.3. The banachic kernel of an inf-compact potential.

**Definition 1.** A **potential** on  $\mathcal{E}$  is a convex, even functionnal vanishing at zero. A potential  $\Phi$  on  $\mathcal{E}$  is said  $\sigma - \inf - \text{compact}$  if there exists  $\lambda \in \mathbb{R}_+^*$  such that :

$$S_\lambda(\Phi) = \{e \in \mathcal{E} ; \Phi(e) \leq \lambda\} \text{ is compact in } (\mathcal{E}, \sigma(\mathcal{E}, \mathcal{E}'))$$

A banachic kernel  $B_\Phi$  of a  $\sigma - \inf - \text{compact}$  potential  $\Phi$  is the subdifferential of the Legendre-Fenchel transform  $\Phi^*$  of  $\Phi$  .

So :  $B_\Phi = \partial\Phi^*$ .

#### Example 1.

A hilbertian kernel is a Banachic kernel of a convenient potential.

#### Example 2.

$$\forall e \in \mathcal{E}, \Phi(e) = \begin{cases} \omega(\|x\|) & \text{if } e = jx, x \in \mathcal{B} \\ +\infty & \text{otherwise} \end{cases}$$

where  $\omega$  is a map from  $\mathbb{R}_+$  into  $\mathbb{R}_+$  derivable, even and such that :  $\omega(0) = 0$  .

**Proposition 1.** Let  $\Phi$  a potential on  $\mathcal{E}$  and  $E_\Phi$  the vectorial subspace of  $\mathcal{E}$  generated by  $\text{dom}\Phi = \{e \in \mathcal{E} ; \Phi(e) < +\infty\}$  . We set :

$$\forall x \in E_\Phi , p_\Phi(x) = \inf \left\{ \lambda \in \mathbb{R}_+^* : \Phi\left(\frac{x}{\lambda}\right) \leq 1 \right\}$$

Then  $p_\Phi$  is a (semi-)norm on  $E_\Phi$  .

**Definition 2.** We say that  $\Phi$  is reflexive if  $(E_\Phi , p_\Phi)$  is a reflexive normed space.

**Proposition 2.**  $\Phi$  is reflexive iff the (convex) l.c.s. regularized of  $\Phi$  is  $\sigma - \inf - \text{compact}$  on  $\mathcal{E}$  .

## 2. FIRST APPLICATION

Let

$$a \in \mathbb{R}_+^*, m \in \mathbb{N}^*, p \in ]1, \infty[ \text{ and } p^* \text{ such that : } \frac{1}{p} + \frac{1}{p^*} = 1$$

We suppose that :

$$\mathcal{E} = \mathbb{R}^{(-a, a)} . \text{ So :}$$

$$\mathcal{E}' = \left\{ \sum_{j \in I} \gamma_j \delta_{t_j} ; \gamma_j \in \mathbb{R} , t_j \in (-a, a) , \text{Card}(I) < +\infty \right\}$$

2.1. **The space  $\mathcal{A}_p$ .** Let us consider the following space :

$$\mathcal{A}_p = \left\{ x \in \mathcal{E} ; \int_{-a}^a |x^{(j)}(t)|^p dt < +\infty, j = 0, 1, \dots, m \right\}$$

Then :

$$\mathcal{C}^m(-a, a) \subset \mathcal{A}_p \subset \mathcal{C}^{m-1}(-a, a)$$

Given  $e'_0, \dots, e'_{m-1} \in \mathcal{E}'$ ,

$$\forall k \in \{0, 1, \dots, (m-1)\}, \forall x \in \mathcal{A}_p, \lambda_k(x) = \langle jx, e'_k \rangle$$

Now, we denote by  $\mathcal{P}_{m-1}(-a, a)$ , the space of polynomials on  $(-a, a)$ , with degree less or equal to  $(m-1)$ , and we suppose that the restrictions of  $\lambda_0, \lambda_1, \dots, \lambda_{(m-1)}$  on  $\mathcal{P}_{m-1}(-a, a)$  are linearly independent. Then we set :

$$\forall x \in \mathcal{A}_p, \|x\|_{\mathcal{A}_p} = \left( \sum_{k=0}^{m-1} |\lambda_k(x)|^p + \int_{-a}^a |x^{(m)}(t)|^p dt \right)^{\frac{1}{p}}$$

and :

$$\forall x \in \mathcal{A}_p, \|x\|_p = \left( \sum_{j=0}^m \left( \int_{-a}^a |x^{(j)}(t)|^p dt \right) \right)^{\frac{1}{p}}$$

Then :  $\|\cdot\|_p$  is a norm on  $\mathcal{A}_p$  and  $(\mathcal{A}_p, \|\cdot\|_p)$  is a Banach space.

**Proposition 3.** *If the functional  $\lambda_0, \lambda_1, \dots, \lambda_{(m-1)}$  are (linear and) continuous on  $(\mathcal{A}_p, \|\cdot\|_p)$*

(i) *The two norms  $\|\cdot\|_p$  and  $\|\cdot\|_{\mathcal{A}_p}$  are equivalent (from the classical Banach theorem)*

*So,  $(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p})$  is a Banach space.*

(ii) *There exists  $m$  polynomials  $P_0, P_1, \dots, P_{(m-1)} \in \mathcal{P}_{m-1}(-a, a)$  such that :*

$$\lambda_k(P_l) = \delta_{kl}, k, l \in \{0, 1, \dots, (m-1)\}$$

Moreover, we know that :

$$\forall x \in \mathcal{C}^m(-a, a), \forall s \in (-a, a), x(s) = \sum_{j=0}^{m-1} \frac{s^j}{j!} + \left( \int_0^s \frac{(s-t)_+^{m-1}}{(m-1)!} \cdot x^{(m)}(t) dt \right)$$

Thus, we can prove that :

$$\forall x \in \mathcal{C}^m(-a, a), \forall s \in (-a, a), x(s) = \sum_{k=0}^{m-1} \lambda_k(x) \cdot P_k(s) + \left( \int_{-a}^a \Lambda_m(s, t) \cdot x^{(m)}(t) dt \right)$$

As  $\mathcal{C}^m(-a, a)$  is dense in  $(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p})$ , the previous formula is true for any  $x \in \mathcal{A}_p$ , and the following proposition is true :

**Proposition 4.**

$$(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p}) \in Ban(\mathcal{E})$$

Let us set :

$$\mathcal{R}_p = \mathcal{P}_{m-1}(-a, a) \text{ and } \forall P \in \mathcal{R}_p, \|P\|_{\mathcal{R}_p} = \left( \sum_{k=0}^{m-1} |\lambda_k(P)|^p \right)^{\frac{1}{p}}$$

Then  $\|\cdot\|_{\mathcal{R}_p}$  is a norm on  $\mathcal{R}_p$ .

So,  $\mathcal{A}_p$  is the algebraic direct sum of  $\mathcal{R}_p$  and :

$$\mathcal{C}_p = \left\{ x \in \mathcal{E} ; \int_{-a}^a |x^{(m)}(t)|^p dt < +\infty \text{ with } \lambda_k(x) = 0, k \in \{0, 1, \dots, (m-1)\} \right\}$$

Let us set :

$$\forall x \in \mathcal{C}_p, \|x\|_{\mathcal{C}_p} = \left( \int_{-a}^a |x^{(m)}(t)|^p dt \right)^{\frac{1}{p}}$$

Then:

$$(\mathcal{C}_p, \|\cdot\|_{\mathcal{C}_p}) \text{ is a Banach space}$$

As  $\dim(\mathcal{R}_p)$  is finite,

$$(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p}) \text{ is the direct topological sum of } (\mathcal{R}_p, \|\cdot\|_{\mathcal{R}_p}) \text{ and } (\mathcal{C}_p, \|\cdot\|_{\mathcal{C}_p})$$

**2.2. The banachic kernel of  $(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p})$ .** (\*) Given  $\mathcal{B} \in \text{Ban}(\mathcal{E})$ , we set :

$$\forall x \in \mathcal{E}, f_{\mathcal{B}}(e) = \begin{cases} \omega_{\mathcal{B}}(\|\cdot\|_{\mathcal{B}}) & \text{if } e = jx, x \in \mathcal{B} \\ +\infty & \text{otherwise} \end{cases}$$

In the following,  $\mathcal{B}$  will be identified to  $\mathcal{A}_p$ ,  $\mathcal{R}_p$  and  $\mathcal{C}_p$  and we suppose that :

$$\omega_{\mathcal{B}}(\rho) = \rho^p \text{ when } \rho \in \mathbb{R}^*$$

Let us set :

$$\forall \rho \in \mathbb{R}^*, \alpha_p(\rho) = |\rho|^{p-1} \cdot \text{signe}(\rho) \implies (\alpha_p)^{-1} = \alpha_{p^*}$$

Then :

$$\begin{aligned} \forall x, y \in \mathcal{A}_p, \forall \mu \in \mathbb{R}, & \left[ \frac{\partial}{\partial \mu} (f(y + \mu x)) \right]_{\mu=0} \\ &= \sum_{k=0}^{m-1} \alpha_p \cdot \lambda_k(y) \cdot \lambda_k(x) + \int_{-a}^a \alpha_p \cdot y^{(m)}(t) \cdot x^{(m)}(t) dt \end{aligned}$$

Let  $A_p$  (resp.  $R_p$  and  $C_p$ ) the banachic kernel of  $(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p})$

(resp.  $(\mathcal{R}_p, \|\cdot\|_{\mathcal{R}_p})$ ,  $(\mathcal{C}_p, \|\cdot\|_{\mathcal{C}_p})$ ) relatively to  $f_{\mathcal{A}_p}$  (resp.  $f_{\mathcal{R}_p}$ ,  $f_{\mathcal{C}_p}$ ).

But,  $\forall x \in \mathcal{A}_p$ ,  $x = x_1 + x_2$ ,  $x_1 \in \mathcal{R}_p$ ,  $x_2 \in \mathcal{C}_p$  (this decomposition is unique).

We deduce that :

$$\forall x \in \mathcal{A}_p, f_{\mathcal{A}_p}(x) = f_{\mathcal{R}_p}(x) + f_{\mathcal{C}_p}(x)$$

**Proposition 5.** (i)

$$\begin{aligned} \forall e', f' \in \mathcal{E}', \quad \langle \mathcal{R}_p e', f' \rangle &= \sum_{k=0}^{m-1} \alpha_{p^*} \langle j P_k, e' \rangle \cdot \langle j P_k, f' \rangle \\ \langle \mathcal{C}_p e', f' \rangle &= \int_{-a}^a \langle j \Lambda_m(\cdot, \theta), f' \rangle \cdot \alpha_{p^*} \langle j \Lambda_m(\cdot, \theta), e' \rangle d\theta \end{aligned}$$

(ii)  $A_p = R_p + C_p$  .  
So :

$$\forall s, t \in (-a, a) \quad pp, \quad \langle A_p \delta_s, \delta_t \rangle = \sum_{k=0}^{m-1} P_k(t) \cdot \alpha_{p^*} P_k(s) + \int_{-a}^a \Lambda_m(t, \theta) \alpha_{p^*} \Lambda_m(s, \theta) d\theta$$

**Remark 1.**

One can verify that :  $A_2, R_2, C_2$  are hilbertian kernels.

### 2.3. Some properties of $C_p$ .

(i) Let  $a \in R_+^*$ ,  $p, q \in ]1, \infty[$  with  $p < q$  ; then :  $\mathcal{C}_q \subset \mathcal{C}_p$  and  $\mathcal{C}_q$  is dense in  $\mathcal{C}_p$  .

Now, we set :

$$\forall s, t \in (-a, a) \quad pp, \quad C_p(t, s) = \langle C_p \delta_s, \delta_t \rangle$$

From previous results we deduce that :

$$\forall s, t \in (-a, a) \quad pp, \quad \frac{\partial^m}{\partial t^m} (\mathcal{C}_q(t, s)) = \alpha_{1+\frac{p-1}{q-1}} \left( \frac{\partial^m}{\partial t^m} (\mathcal{C}_p(t, s)) \right)$$

This formula is symmetric relatively to  $p$  and  $q$  because :

$$\left( \alpha_{1+\frac{p-1}{q-1}} \right)^{-1} = \alpha_{1+\frac{q-1}{p-1}}$$

Thus, this formula is true for any  $p, q \in ]1, \infty[$  .

So, when  $p = 2$  , we have :

$$\forall q \in ]1, \infty[ , \forall s, t \in (-a, a) \quad pp, \quad \frac{\partial^m}{\partial t^m} (\mathcal{C}_q(t, s)) = \alpha_{q^*} \left( \frac{\partial^m}{\partial t^m} (\mathcal{C}_2(t, s)) \right) ,$$

$$\frac{1}{q} + \frac{1}{q^*} = 1$$

As,  $\alpha_p + \alpha_{p^*} = 1$  and as  $\mathcal{C}_2$  is a hilbertian kernel, therefore a symmetric kernel, we have :

$$\begin{aligned} \forall s, \theta \in (-a, a) \quad pp, \\ \mathcal{C}_p(\theta, s) &= \int_{-a}^a \left( \frac{\partial^m}{\partial t^m} (\mathcal{C}_2(t, s)) \right) \alpha_{p^*} \left( \frac{\partial^m}{\partial t^m} (\mathcal{C}_2(\theta, t)) \right) dt \\ \text{with : } \mathcal{C}_p(s, s) &= \int_{-a}^a \left( \frac{\partial^m}{\partial t^m} (\mathcal{C}_2(t, s)) \right)^{p^*} dt \end{aligned}$$

(ii) When  $\mathcal{E} = \mathcal{D}'(-a, a)$  and  $\mathcal{E}' = \mathcal{D}(-a, a)$ , we have :

$$\begin{aligned} \forall t &\in (-a, a) \text{ pp}, \forall \varphi, \psi \in \mathcal{E}, \\ (*) \quad (-1)^m \alpha_p \left( \frac{\partial^m}{\partial t^m} (\tilde{\mathcal{C}}_p \varphi)(t) \right) &= \varphi(t) \text{ with :} \\ \lambda_k (\tilde{\mathcal{C}}_p \varphi) &= 0, k = 0, 1, \dots, (m-1) \end{aligned}$$

$$\begin{aligned} (**) \quad \langle \mathcal{C}_p \psi, \varphi \rangle &= \int_{-a}^a (\tilde{\mathcal{C}}_p \psi)(t) \varphi(t) dt \\ &= \int_{-a}^a \left( \frac{\partial^m}{\partial t^m} (\tilde{\mathcal{C}}_2 \varphi)(t) \right) \alpha_{p^*} \left( \frac{\partial^m}{\partial t^m} (\tilde{\mathcal{C}}_2 \psi)(t) \right) dt \end{aligned}$$

and :

$$\langle \mathcal{C}_p \varphi, \varphi \rangle = \int_{-a}^a \left( \frac{\partial^m}{\partial t^m} (\tilde{\mathcal{C}}_2 \varphi)(t) \right)^{p^*} dt$$

**2.4. Banachic kernel and the Sard's factorization theorem.** Let  $D^m$  the derivative of order  $m$ ; that is a linear and continuous mapping from  $(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p})$  onto  $L^p(a, b)$ .

Given  $e' \in \mathcal{E}'$ , we shall denote  $v_{e'}$  the mapping from  $\mathcal{A}_p$  into  $\mathbb{R}$  such that :

$$\forall x \in \mathcal{A}_p, v_{e'}(x) = \left\langle j \left( x - \sum_{k=0}^{m-1} \lambda_k(x) . P_k \right), e' \right\rangle$$

So, we have the following scheme :

$$\begin{array}{ccc} & \mathcal{A}_p & \\ & \swarrow \searrow & \\ L^p(a, b) & \longrightarrow & \mathbb{R} \end{array}$$

From the Sard's factorization theorem we deduce that there exists  $G \in L^{p^*}(a, b)$  such that :

$$v_{e'}(x) = \int_a^b G(t) . x^{(m)}(t) dt$$

thus :

$$G = \alpha_p \left( \left( \tilde{\mathcal{C}}_p e' \right)^{(m)} \right)$$

### 3. BANACHIC B-SPLINES

**3.1. About an approximation problem in a Banach space.** Let  $n \in \mathbb{N}$ ,  $e'_0, e'_1, \dots, e'_n$ ,  $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}$  and :

$$\Gamma = \{e \in \mathcal{E} ; \langle e, e'_k \rangle = \alpha_k, k = 0, 1, \dots, n\}$$

Then  $\Gamma$  is an hyperplane (with finite codimension) which is closed in  $(\mathcal{E}, \sigma(\mathcal{E}, \mathcal{E}'))$ . Let  $\Phi$  be a  $\sigma$ -inf-compact potential on  $\mathcal{E}$ . Let us consider the following problem :

$$\Pi : \inf \{ \Phi(e) ; e \in \Gamma \}$$

**Proposition 6.** *Let us suppose that :*

$$\inf \{ \Phi(e) ; e \in \Gamma \} = \mu \in \mathbb{R}$$

*Then  $\Pi$  has a solution (which is unique when  $\Phi$  is strictly convex).*

### 3.2. Characterization of a solution $\bar{e}$ of $\Pi$ .

**Theorem 1.** *Let  $\Gamma_0(\mathcal{E})$  be the set of proper convex l.c.s. functional on  $\mathcal{E}$ , which are not identical at  $+\infty$ ,  $\Delta$  a convex subset of  $\mathcal{E}$  and  $u \in \Gamma_0(\mathcal{E})$ .*

*Let us consider the following problem :*

$$\Pi : \inf \{ u(e) ; e \in \Delta \}$$

*If the following hypotheses are verified :*

- (\*)  $\inf \{ u(e) ; e \in \Delta \} = \mu \in \mathbb{R}$
- (\*\*)  *$u$  is finite and continuous at a point of  $\Delta$   
or  $u$  is finite at an interior point of  $\Delta$*

*then :*

(i)

$$\begin{aligned} \mu &= \max \{ -u^*(-e') - h(e'; \bar{\Delta}) ; e' \in \text{dom}(h(\cdot; \bar{\Delta})) \} \\ \text{where } u^* &\text{ is the dual of } u, \bar{\Delta} \text{ is the closure of } \Delta \text{ in } (\mathcal{E}, \sigma(\mathcal{E}, \mathcal{E}')) \\ \text{and } h(\cdot; \bar{\Delta}) &= (\delta(\cdot; \bar{\Delta}))^* \end{aligned}$$

(ii) *Moreover, if*

$$\{ e \in \mathcal{E} ; u(e) + \delta(e; \Delta) \leq \mu \}$$

*is a non-void set*

*or, if*

$$\exists e'_0 \in \mathcal{E}' ; \partial u^*(-e'_0) \cap \partial h(e'_0; \bar{\Delta})$$

*is a non-void set,*

*then :*

$$\begin{aligned} &\{ e \in \mathcal{E} ; u(e) + \delta(e; \Delta) \leq \mu \} \\ &= \cup \{ (\partial u^*(-e') \cap \partial h(e'; \bar{\Delta})) ; e' \in \mathcal{E}' ; \} \end{aligned}$$

**Proposition 7.** *Let us suppose that  $\Phi$  is finite and continuous at a point in  $\Gamma$ .*

*Let  $\bar{e}$  a solution of  $\Pi$ . Then :*

$$\bar{e} \in B_\Phi \left( \sum_{j=0}^n \bar{\lambda}_j e'_j \right) \text{ where } \bar{\lambda}_0, \bar{\lambda}_1, \dots, \bar{\lambda}_n \in \mathbb{R} \text{ and } \bar{e} \in \Delta \text{ and } B_\Phi = \partial \Phi^*$$

**3.3. An application.** I shall use below the same hypotheses and notations as in the previous paragraphs.

Given  $e'_0, \dots, e'_{m-1}, e'_m, \dots, e'_n \in \mathcal{E}'$  and  $a_0, \dots, a_m, a_{m-1}, \dots, a_n \in \mathbb{R}$ , we consider the following problem :

$$\Pi_0 : \inf \left\{ ; \int_{-a}^a |x^{(m)}(t)|^p dt ; x \in \mathcal{A}_p, \langle jx, e'_l \rangle = \alpha_l, l \in \{0, \dots, n\} \right\}$$

$\Pi_0$  is equivalent to :

$$\Pi_1 : \inf \left\{ ; \int_{-a}^a |x^{(m)}(t)|^p dt + \sum_{k=0}^{m-1} |\lambda_k(x)|^p ; x \in \mathcal{A}_p, \langle jx, e'_l \rangle = \alpha_l, l \in \{0, \dots, n\} \right\}$$

Thus, we deduce that :

$$\sigma_p = \tilde{A} \left( \sum_{l=0}^n \mu_l e'_l \right)_p ; \mu_0, \mu_1, \dots, \mu_n \in \mathbb{R} \text{ with } \langle j\sigma_p, e'_l \rangle = \alpha_l, l \in \{0, \dots, n\}$$

So :

$$\begin{aligned} \forall t &\in (-a, a) \text{ pp}, \\ \sigma_p(t) &= \sum_{k=0}^{m-1} P_k(t) \cdot \alpha_{p^*} \left[ \sum_{l=0}^n \mu_l \langle jP_k, e'_l \rangle \right] \\ &\quad + \int_{-a}^a \Lambda_m(t, \theta) \alpha_{p^*} \left[ \sum_{l=0}^n \mu_l \langle j\Lambda_m(\cdot, \theta), e'_l \rangle \right] d\theta \\ \text{with } \langle j\sigma_p, e'_l \rangle &= \alpha_l, l \in \{0, \dots, n\} \end{aligned}$$

Then, to calculate  $\mu_0, \mu_1, \dots, \mu_n$  we must solve a non-linear system of equations.

**Properties of  $\sigma_p$  :**

Let  $\tau_2$  the solution of the following problem :

$$\Pi_2 : \inf \left\{ ; \int_{-a}^a \left| x^{(m)}(t) \right|^2 dt ; x \in \mathcal{A}_2, \langle jx, e'_l \rangle = \beta_l, l \in \{0, \dots, n\} \right\}$$

Then :

$$\begin{aligned} \forall t &\in (-a, a) \text{ pp}, \\ \tau_2(t) &= \sum_{k=0}^n \nu_k \sum_{l=0}^{m-1} P_l(t) \cdot \langle jP_l, e'_k \rangle \\ &\quad + \int_{-a}^a \Lambda_m(t, \theta) \sum_{l=0}^n \langle j\Lambda_m(\cdot, \theta), e'_k \rangle d\theta \\ \text{with } \langle j\tau_2, e'_l \rangle &= \beta_l, l \in \{0, \dots, n\} \end{aligned}$$

Let us suppose that we have choose  $\beta_0, \dots, \beta_n \in \mathbb{R}$ , such that :

$\nu_l = \mu_l, l = 0, 1, \dots, n$ . Then :

$$\begin{aligned} \forall t &\in (-a, a) \text{ pp}, \\ \tau_2^{(m)}(t) &= \sum_{l=0}^n \mu_l \langle j\Lambda_m(\cdot, t), e'_l \rangle = \alpha_p \left[ \sigma^{(m)}(t) \right] \end{aligned}$$

Thus :

$$\sigma_p^{(m)}(t) = \alpha_{p^*} \left( \tau_2^{(m)}(t) \right)$$

So, if  $e'_l = \delta_{t_l}, t_l \in (-a, a), l = 0, 1, \dots, n$ , we have :

$$\sigma_p^{(m)}(t) = \alpha_{p^*} \left( \sum_{l=0}^n \mu_l \Lambda_m(t_l, t) \right)$$



**3.4. Banachic B-splines.** Let  $L_{k,m-1}$  be the polynomial with degree less or equal to  $(m-1)$  such that :

$$L_{k,m-1}(t_l) = \delta_{k,l} \quad , \quad k, l = 0, 1, \dots, (m-1)$$

Then :

$$\begin{aligned} \forall s, t &\in (-a, a) \text{ pp} \\ \Lambda_m(s, t) &= \frac{1}{(m-1)!} \left[ (s-t)_+^{m-1} - \sum_{k=0}^n (t_k - t)_+^{m-1} L_{k,m-1}(s) \right] \end{aligned}$$

Now, let us suppose that  $n \geq m$ .

Let us choose  $\mu_0, \dots, \mu_n \in \mathbb{R}$ , such that :

$\sum_{l=0}^n \mu_l f(t_l)$  is equal to the divided difference of order  $m$ , at the point  $\tau \in (-a, a)$

So :

$$\begin{aligned} \forall t &\in (-a, a) \text{ pp} , \\ \tau_2^{(m)}(t) &= \sum_{l=0}^n \mu_l \frac{1}{(m-1)!} \left[ (t_l - t)_+^{m-1} - \sum_{k=0}^n (t_k - t)_+^{m-1} L_{k,m-1}(t_l) \right] \\ &= \sum_{l=0}^n \mu_l \frac{(t_l - t)_+^{m-1}}{(m-1)!} \end{aligned}$$

Thus :

$\tau_2^{(m)}$  is a classical polynomial B-spline of degree equal at  $(m-1)$ .

The B-spline  $\tau_2^{(m)}$  is equal to zero on the set  $(-\infty, a_1) \cup (a_2, +\infty)$  with  $a_1, a_2 \in (-a, a)$  and  $a_1 < a_2$ .

(We remark moreover that  $\tau_2^{(m)}$  is an optimization spline function.)

As  $\alpha_{p^*} = 0$ , because  $p^* > 1$ ,

$\sigma_p^{(m)}$  is equal to zero on the set  $(-\infty, a_1) \cup (a_2, +\infty)$ .

**Definition 3.** I say that  $\sigma_p^{(m)}$  is a banachic B-spline.

Let us suppose that :  $t_{l+1} - t_l = h$ ,  $l = 0, 1, \dots, (n-1)$  ;

Let us set :

$$D_h^m f = \sum_{l=0}^n \mu_l f(t_l) \quad \text{and} \quad Q_h^{m,2} = \tau_2^{(m)}, \quad Q_h^{m,p} = \sigma_p^{(m)}$$

Then :

$$Q_h^{m,2} = \alpha_p(Q_h^{m,p}) \quad \text{with :} \quad \alpha_p(\rho) = |\rho|^{p-1} \text{sign}(\rho)$$

From the classical properties of  $Q_h^{m,2}$ , we deduce easily those of  $Q_h^{m,p}$ . So,

**Proposition 8.**

- (i)  $0 \leq Q_h^{m,p}(s, t) \leq h^{1-p}$
- (ii)  $\int_{\mathbb{R}} (Q_h^{m,p}(s, t))^{p-1} ds = 1$  and  $\sum_{j \in \mathbb{Z}} (Q_h^{m,p}(jh, t))^{p-1} ds = h^{-1}$
- (iii)  $\forall y \in \mathcal{C}^m(\mathbb{R})$ ,  $\int_{\mathbb{R}} y^{(m)}(s) (Q_h^{m,p}(s, t))^{p-1} ds = D_h^m y(t)$

Other properties can be found in the book of the author :

"Hilbertian kernels and spline functions".

## 4. EXTENSIONS

- 4.1. \*  $M$  is a potential such that  $M^*$  is derivable or  
 when  $M$  is convex, positive, such that  $M(0) = 0$  and  $M^*$  is derivable,  
 we can consider the case where :

$$\frac{\partial^m}{\partial t^m} \left( \tilde{C}_p e' \right) (t) = (M^*)' (\langle j \Lambda_m(\cdot, t), e' \rangle)$$

Then, what space is associated to  $M$  ?

- 4.2. **Calculus.** of the banachic kernels of the spaces  $\mathcal{B}_{p,k}$  when  $p \in ]1, +\infty[$   
 where  $\mathcal{B}_{p,k}$  is the space of distributions  $x \in S'(\mathbb{R})$  such that :

$$\|x\|_{p,k} = \left[ (2\pi)^{-n} \int_{\mathbb{R}^n} |k(\zeta) \cdot \hat{x}(\zeta)|^p d\zeta \right]^{\frac{1}{p}}$$

$k$  is a mapping from  $\mathbb{R}^n$  into  $\mathbb{R}_+$  such that :

$$\forall \zeta, \eta \in \mathbb{R}^n, k(\zeta + \eta) \leq (1 + c|\zeta|^\nu) \cdot k(\eta)$$

There are also many other extensions to banachic kernels of the properties of hilbertian kernels (wavelets, fractals, vectorial case, etc.)

#### 5. BANACHIC KERNELS AND PARTIAL DIFFERENTIAL EQUATIONS. AN EXAMPLE.

Let  $K$  a triangle in  $\mathbb{R}^2$  which vertices are  $A = (0,0)$  ,  $B = (1,0)$  ,  $C = (0,1)$  .

We denote by  $\text{int}(K)$  the interior of  $K$  .

Let  $\mathcal{E} = \mathcal{D}'(\text{int}(K))$  and  $\mathcal{E}' = \mathcal{D}(\text{int}(K))$  .

Given  $p \in ]1, +\infty[$  , let us consider the Banach space  $\mathcal{B}$  of continuous functions on  $K$  which are equal to zero at  $A, B, C$  embedded with the following norm :

$$v \rightarrow \sum_{|\alpha|=1} \left( \int_K |D^\alpha v(t)|^p dt \right)^{\frac{1}{p}}$$

Then , using the same notations as those which are defined at the paragraph 1.2,

we have :

$$\begin{aligned} \forall \varphi &\in \mathcal{E}', \forall v \in \mathcal{B}, \\ \langle jv, \varphi \rangle &= \left\| \tilde{B}\varphi \right\|^{p-1} \cdot \left( \frac{d}{d\lambda} \left( \left\| \tilde{B}(\varphi + \lambda v) \right\| \right) \right)_{\lambda=0} \end{aligned}$$

But :

$$\frac{d}{d\lambda} (\|u + \lambda v\|) = \|u\|^{1-p} \cdot \sum_{|\beta|=1} \left( \int_K |D^\beta u(t)|^{p-1} \cdot \text{sign}(D^\beta u(t)) \cdot D^\beta v(t) dt \right)$$

Now, we set :

$$\forall u \in \mathcal{E}, Lu = \sum_{|\beta|=1} \left( \int_K D^\beta \left( |D^\beta u|^{p-1} \cdot \text{sign}(D^\beta u) \right) \right) \cdot$$

So, the banachic kernel  $B$  of  $\mathcal{B}$  is the solution of the following non linear differential

problem :

$$\forall \varphi \in \mathcal{E}', L(B\varphi) = \varphi \text{ with } (B\varphi)(A) = (B\varphi)(B) = (B\varphi)(C) = 0$$

**References.**

- [1] M. Atteia, J. Audounet. inf compact potentials and banachic kernels.  
Lectures notes in Mathematics, 991, pages 7–27, 1983.  
*E-mail address:* `marcatteia@orange.fr`